The transformations of the electromagnetic potentials under translations

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I consider infinitesimal translations $x'^{\alpha} = x^{\alpha} + \delta x^{\alpha}$ and demand that Noether's approach gives a symmetric electromagnetic energy-momentum tensor as it is required for gravitational sources. This argument determines the transformations of the electromagnetic potentials under infinitesimal translations to be $A'_{\gamma}(x') = A_{\gamma}(x) + \partial_{\gamma}[\delta x_{\beta}\, A^{\beta}(x)]$, which differs from the usually assumed invariance $A'_{\gamma}(x') = A_{\gamma}(x)$, by the gauge transformation $\partial_{\gamma}[\delta x_{\beta}\, A^{\beta}(x)]$.

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In relativistic field theory it is well known, and often referred to as Noether's theorem [1], that each independent infinitesimal symmetry implies a conserved current with an associated constant or motion. Here we are interested in the symmetry under infinitesimal translations

$$x'^{\,\alpha} = x^{\alpha} + \delta x^{\alpha} \tag{1}$$

for which Noether's theorem yields the conserved currents of the energy-momentum tensor with the associated constants of motion being energy and momenta. However, the energy-momentum tensor $T^{\alpha\beta}$ obtained in this way from the electromagnetic Lagrangian

$$\mathcal{L} = -\frac{1}{16\pi}\, F_{\alpha\beta}\, F^{\alpha\beta} \quad \text{with} \quad F^{\alpha\beta} = \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha} \tag{2}$$

is not symmetric, whereas everyone believes that the correct result ought to be symmetric. The standard procedure [2] is to add a total divergence such that the final result becomes the desired symmetric tensor $\theta^{\alpha\beta}$. While this procedure is acceptable within electrodynamics, it becomes questionable as soon as one is concerned about gravity. The electromagnetic energy-momentum distributions of $T^{\alpha\beta}$ and $\theta^{\alpha\beta}$ differ and this changes the implied gravitational force. This is in principle observable [3], although in practice presumably not, unless someone identifies suitable cosmological field distributions. On the theory side, in gravity the symmetry transformations of general covariance yield the symmetric energy-momentum tensor as source of the gravitational field, see for instance [4,5], and this is presumably the strongest evidence underlining that the energy-momentum distribution of $\theta^{\alpha\beta}$ is the correct one.

In view of the arguments in favor of the symmetric energy-momentum tensor it is astonishing that Noether's theorem leads to a tensor $T^{\alpha\beta}$ with an obviously incorrect energy-momentum distribution. In particularly, one should bear in mind that the derivation of $T^{\alpha\beta}$ relies only on the symmetry transformations of a four vector under translations

$$A'_{\gamma}(x') = A_{\gamma}(x) \tag{3}$$

and on factoring out the *local* translation δx^{α}. Due to the locality of the procedure, it is hard to imagine that it could lead to an incorrect energy momentum distribution when the fundamental assumptions are sound.

In this letter I give a simple solution to the problem. The non-symmetric energy momentum tensor $T^{\alpha\beta}$ is obtained under the assumption that A_{γ} transforms as a four vector (3) under translations (1). Due to the gauge invariance of the electromagnetic Lagrangian (2) this does not have to be the case. We may allow for more general transformations which differ from (3) by gauge transformations, *i.e.*

$$A'_{\gamma}(x') = A_{\gamma}(x) + \partial_{\gamma}\, \Lambda(x)\ . \tag{4}$$

That nature may employ such a transformation behavior instead of (3) is not entirely a surprise. On the quantum level the electromagnetic fields rely on superpositions of massless creation and annihilation operators and Weinberg [6] points out to us that such fields do not allow for representations of the (proper) Lorentz group, but only for transformations which differ from those by a gauge transformation. Therefore, it is quite natural to conjecture that nature uses gauge transformation also for translations. Repeating the arguments of Noether's theorem with the ansatz (4) and requesting a symmetric energy-momentum tensor leads to the unique solution

$$A'_{\gamma}(x') = A_{\gamma}(x) + \partial_{\gamma}\left[\delta x_{\beta}\, A^{\beta}(x)\right] \tag{5}$$

which is conjectured to be the transformation law realized by nature for infinitesimal translations of electromagnetic potentials. The remainder of the paper is devoted to the derivation of this equation. Up to some notational changes and adaptions to the case at hand, my arguments follow closely chapter 1 of Bogoliubov and Shirkov [7].

First, let us quickly recall how relativistic field equations are derived from the action principle. The action is a four dimensional integral over a scalar Lagrangian density

$$\mathcal{A} = \int d^4x\, \mathcal{L}(\psi_k,\, \partial_{\alpha}\psi_k) \tag{6}$$

and, therefore, by itself a scalar under the connected part of the Lorentz group. Variations of the fields are defined as functions

$$\delta\psi_k(x) = \psi'_k(x) - \psi_k(x) \tag{7}$$

which are non-zero for some localized space-time region. The action is required to vanish under such variations

$$0 = \delta\mathcal{A} =$$

$$\sum_k \int d^4x \left[(\delta\psi_k)\, \frac{\partial \mathcal{L}}{\partial \psi_k} + (\delta\, \partial_\alpha \psi_k)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_k)} \right] . \tag{8}$$

Integration by parts gives

$$0 = \sum_k \int d^4x\, (\delta\psi_k) \left[\frac{\partial \mathcal{L}}{\partial \psi_k} - \partial_\alpha\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_k)} \right] , \tag{9}$$

where we used that the surface terms vanish. As the variations $\delta\psi_k$ are independent, the integrand in (9) has to vanish for each k and we arrive at the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \psi_k} - \partial_\alpha\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_k)} = 0 \tag{10}$$

for relativistic fields. For the electrodynamic Lagrangian (2) they yield $\partial_\alpha\, F^{\alpha\beta} = 0$.

Noether's theorem applies to transformations of the coordinates for which the transformations of the field functions are also known. Such transformations constitute a symmetry of the theory when the corresponding variation of the action vanishes. The theorem states that to each such symmetry a combination of the field functions exists which defines a conserved current. For this purpose we introduce, in addition to (7), a second type of variations which combines space-time and their corresponding field variations

$$\overline{\delta}\psi_k(x) = \psi'_k(x') - \psi_k(x)\,. \tag{11}$$

Using (note $\delta x^\alpha\, \partial_\alpha \psi'_k = \delta x^\alpha\, \partial_\alpha \psi_k$ because δ^2 variations disappear)

$$\psi'_k(x') = \psi'_k(x) + \delta x^\alpha\;\partial_\alpha \psi_k(x)$$

we find a relation between the variations (11) and (7)

$$\overline{\delta}\psi_k(x) = \delta\psi_k(x) + \delta x^\alpha\;\partial_\alpha \psi_k(x)\,. \tag{12}$$

For a scalar field ψ (as well as for ordinary four vector fields) symmetry under translations means

$$\overline{\delta}\psi(x) = \psi'(x') - \psi(x) = 0\,. \tag{13}$$

But for the electromagnetic potentials we allow (4)

$$\overline{\delta}A_\gamma(x) = A'_\gamma(x') - A_\gamma(x) = \partial_\gamma \Lambda(x)\,. \tag{14}$$

With these symmetries equation (12) reduces for a scalar field to

$$\delta\psi = -\delta x^\alpha\;\partial_\alpha \psi(x) \tag{15}$$

and for the electromagnetic potentials to

$$\delta A_\gamma = \partial_\gamma \Lambda(x) - \delta x^\alpha\;\partial_\alpha A_\gamma(x)\ . \tag{16}$$

As the Lagrange density is a scalar, we get for its combined variation (11)

$$0 = \overline{\delta}\mathcal{L} = \mathcal{L}'(x') - \mathcal{L}(x) = \delta\mathcal{L} + \delta x^\alpha\;\partial_\alpha \mathcal{L} \tag{17}$$

where besides (13) we used the relation (12). Our aim is to factor an over-all variation δx^α out. For $\delta\mathcal{L}$ we proceed as in equation (8), where the fields ψ_k are now replaced by the gauge potentials A_γ

$$\delta\mathcal{L} = (\delta A_\gamma)\, \frac{\partial \mathcal{L}}{\partial A_\gamma} + (\delta \partial_\alpha A_\gamma)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)}\,.$$

Using the Euler-Lagrange equation (10) to eliminate $\partial\mathcal{L}/\partial A_\gamma$, we get (the calculation remains valid in our case where $\mathcal{L}$ does not depend on A_γ)

$$\delta\mathcal{L} = (\delta A_\gamma)\, \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)} + (\delta \partial_\alpha A_\gamma)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)}$$

$$= \partial_\alpha \left[(\delta A_\gamma)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)} \right] .$$

Let us collect all terms which contribute to $\overline{\delta}\mathcal{L}$ in equation (17). For this, note that $\partial_\beta \delta x^\alpha = 0$ holds for all combinations of indices α, β. (Namely, for $\alpha = \beta$ we are led to $\delta 1 = 0$ and for $\beta \neq \alpha$ the variations δx^α are then independent of the coordinates x^β). We find

$$0 = \overline{\delta}\mathcal{L} = \partial_\alpha\, \left[(\delta A_\gamma)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_{\gamma)}} + \delta x^\alpha\;\mathcal{L} \right] =$$

$$\partial_\alpha\, \left[\left(\partial_\gamma \Lambda(x) - (\delta x_\beta)\, \partial^\beta A_\gamma\right)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)} + g^{\alpha\beta}\, \delta x_\beta\, \mathcal{L} \right]$$

where equation (16) was used for the last step. To be able to factor δx_β out of the bracket, one has to request

$$\Lambda(x) = \delta x_\beta\, B^\beta(x) \tag{18}$$

where $B^\beta(x)$ is a not yet determined potential field. With this we get

$$0 = \delta x_\beta\, \partial_\alpha\, \left[(\partial^\beta A_\gamma - \partial_\gamma B^\beta)\, \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)} - g^{\alpha\beta}\, \mathcal{L} \right] .$$

As the variations δx_β are independent, the energy-momentum tensor

$$\theta^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\gamma)}\; (\partial^\beta A_\gamma - \partial_\gamma B^\beta) - g^{\alpha\beta}\, \mathcal{L} \tag{19}$$

gives the conserved currents

$$\partial_\alpha\, \theta^{\alpha\beta} = 0\,. \tag{20}$$

Let us demand that the energy-momentum tensor (19) is symmetric. This leads to the requirement

$$B^{\beta}(x) = A^{\beta}(x) \tag{21}$$

for which

$$\theta^{\alpha\beta} = \frac{1}{4\pi}\left(F^{\alpha\gamma}\,F_{\gamma}{}^{\beta} + \frac{1}{4}g^{\alpha\beta}\,F_{\gamma\delta}\,F^{\gamma\delta}\right) \tag{22}$$

is symmetric because of

$$F^{\alpha\gamma}\,F_{\gamma}{}^{\beta} = F^{\beta\gamma}\,F_{\gamma}{}^{\alpha}\ .$$

Indeed, equation (22) is the symmetric tensor of the textbooks [2]. It differs from other versions of (19) by total divergencies.

In conclusion, I have presented an argument in favor of the transformation behavior (5) and it appears that the question [3] of the energy-momentum distribution of the electromagnetic field may finally be put at rest with the expected result. Noether's theorem alone has no predictive power about whether the energy-momentum tensor is the symmetric or not, but evrything is consistent in the sense that a transformation law exists which gives the tensor (22). In addition, when symmetry of the tensor is assumed the result for $\theta^{\alpha\beta}$ is unique. It is shown in a forthcoming paper that this approach works also for non-abelian gauge theories.

Note added

After posting this manuscript Prof. Jackiw kindly informed me that my result is a special case of his work [8], see [9] for details. Prof. Hehl communicated that the use of 1-Forms leads directly to a symmetric energy-momentum tensor, see for instance [10].

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